THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050B Mathematical Analysis I (Fall 2016)
Tutorial Questions for 6 Oct

1.

Theorem 1. *(Ratio test)*
Let (x_n) be a sequence of positive real numbers. Suppose

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L,$$

where L is a non-negative real number. Then we have:

(a) If $0 \leq L < 1$, then $\lim_{n\to\infty} x_n = 0$.
(b) If $L > 1$, then (x_n) is divergent.
(c) If $L = 1$, this method is inconclusive.

2. (Comparison of Order of Growth)
Using Ratio test, show that we have the following inequalities:

$$n << n^2 << 2^n << n! << n^n$$

where, e.g. $2^n << n!$ is read as

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0$$

Intuitively, $a_n << b_n$ means that a_n is negligible compared to b_n eventually as n grows to infinity.

3. (Average of a Sequence)

Definition 1. *Let (a_n) be any sequence of real numbers. We define its partial sum*

$$S_n := \sum_{k=1}^{n} a_k,$$

and then the average of it by

$$A_n := \frac{S_n}{n}.$$

(a) Show that if $\lim_{n\to\infty} a_n = l \in \mathbb{R}$, then

$$\lim_{n\to\infty} A_n = l.$$

(b) (Optional) We define the Cesàro sum of a sequence (a_n) by

$$\sigma_n := \frac{S_1 + S_2 + \cdots + S_n}{n},$$

where S_n is the partial sum of a_n. (Hence the Cesàro sum is the average of the partial sums)

Show that as a corollary, σ_n converges to $l \in \mathbb{R}$ if

$$\sum_{k=1}^{\infty} a_k := \lim_{n\to\infty} \sum_{k=1}^{n} a_k = l.$$

(c) Show that the converse of (a) is not true by constructing a real sequence a_n whose average converges to a finite limit $l \in \mathbb{R}$ but a_n itself diverges.

(d) (Optional) Show that the converse of (b) is not true by constructing a real sequence a_n whose Cesàro sum converges to a finite limit $l \in \mathbb{R}$ but its partial sum S_n diverges.